11.2 Entropy of a black hole

This is problem 2.42 on page 84 of the text. We went over in general in class. Here are the specifics of the problem.

(a) By dimensional analysis, the radius must be proportional to GM/c^2.

(b) Ordinarily, the entropy of a system is of the same order as the number of particles in the system. If we take a system of N particles and compress it to form a black hole, the second law requires that when we're done, the entropy of the black hole is still at least of order N. But since the end result is the same whether we start with a lot of particles or a few (with the same total mass), the final entropy must in fact be of the order of the *maximum* N, the largest possible number of particles that it could have been formed from.

(c) Suppose we start with N photons, each of which has a wavelength equal to the size of the black hole: $\lambda = GM/c^2$. Each photon has an energy $E = hc/\lambda$, and the total energy of all of them must equal Mc^2:

$$Mc^2 = NE = \frac{Nhc}{\lambda} = \frac{Nhc^3}{GM}.$$

Solving for N gives

$$N = \frac{GM^2}{hc},$$

and so the entropy in conventional units must be of order

$$S \sim \frac{GM^2k}{hc}.$$

(d) For a one-solar mass black hole,

$$\frac{S}{k} = 1.06x10^{77}; \qquad S = 1.5x10^{54}J/K.$$

This is an enormous entropy. For comparison, an ordinary star like the sun contains something like 10^{57} particles, so its entropy is something like $10^{57}k$. To equal the entropy of a single one-solar- mass black hole, you would need 10^{20} ordinary stars or enough to populate a billion (10^9) Milky Way galaxies. Furthermore, since the entropy of a black hole is proportional to the square of its mass, a million-solar mass black hole (as may exist at the center of our galaxy) would have a trillion times the entropy of a one-solar-mass black hole.

12 Chapter 3: Temperature and Entropy

We've determined (defined) thermal equilibrium to be the condition that arises when two systems are brought into thermal contact and allowed to achieve maximal entropy. Consider two systems A and B with internal energies U_A and U_B, and entropies S_A and S_B. The total entropy of the combined system is $S_T = S_A + S_B$. The maximal entropy condition (2nd law of thermo) holds that the total entropy is maximal with respect to either U_A or U_B:

$$\begin{aligned} \frac{\partial S_T}{\partial U_A} &= 0 \\ \frac{\partial S_A}{\partial U_A} + \frac{\partial S_B}{\partial U_A} &= 0 \end{aligned}$$

Now, the total energy $U = U_A + U_B$ is a constant. This implies that $\partial U_A = -\partial U_B$. So, we have the thermal equilibrium condition:

$$\frac{\partial S_A}{\partial U_A} = \frac{\partial S_B}{\partial U_B}$$

. We will use this condition to define temperature:

$$T^{-1} \equiv \frac{\partial S}{\partial U}\|_{N,V} \tag{73}$$

Usually S is a convex function of U. There are cases where S is no convex in U. We discussed these in class:

- "normal" systems: S is convex in U - $\partial S/\partial U$ decreases as U increases. In this case the temperature increases as U increases.

- "miserly" systems: $\partial S/\partial U$ increases as U increases. In this case, the temperature decreases as U is increased. Examples of such systems include gravitationally bound clusters - stars, galaxies, planetary systems, etc. Adding energy to a particle in orbit around a star will cause the particle to climb up the gravitational well, finding a higher orbit and necessarily slowing down. This reduction in orbital velocity corresponds to a decrease in the effective temperature of the particle. The added U went into gravitational potential energy.
- "enlightened" systems. These are systems for which $\partial S/\partial U$ is negative and grows in magnitude as U increases. The negative $\partial S/\partial U$ means that the effective temperature of the system is negative. This is perfectly acceptable in the context of the current definition of temperature as the inverse of the slope of entropy vs. energy, but makes no sense in the context of the heuristic definition of temperature as the rms kinetic energy per particle. The latter quantity can never be negative. So, there is not a one-to-one correspondence between the two definitions. The only time, we have a disparity is in these "enlightened" systems. Examples of such systems include all models of magnetism in which the individual dipole energies are constrained to a finite set of values. Consider the coin model: Each coin can be heads or tails and nothing else. As the "temperature" is increased, more and more coins flip to their high energy state (heads) until there can be no further flipping. The coins are now in a minimal entropy state even though we've been continually adding energy.

12.1 Measuring Entropies

How much entropy does your thoughtful brain generate in an hour of thinking about entropy? First consider an isothermal thought process: your body temperature is a constant 98.6 °F=310.15 K. Any heat you generate is dissipated into the environment without substantial temperature increase. The heat Q you generate in one hour is your metabolism × 3600 seconds. A thoughtful brain thinking about entropy operates at a metabolic rate of about 70 W. So the total Q in one hour is $Q = 70W \times 3600$ sec. $= 2.52 \times 10^5 J$. For an isothermal process, the entropy generated is

$$\Delta S = \frac{Q}{T} = \frac{2.52 \times 10^5 J}{310.15K} = 812.5 J/K$$

What is the increase in the universe's multiplicity as a result of you thinking about entropy for an hour?

$$\Delta \log \Omega = \frac{\Delta S}{k} = 5.89 \times 10^{25} \Rightarrow \Omega_f = \Omega_i e^{5.89 \times 10^{25}} = \Omega_i \times 10^{2.56 \times 10^{25}}$$

12.2 Heat Capacities

The process for calculating heat capacities is:

1. Quantum Mechanics + Combinatorics is used to generate the multiplicity function Ω.
2. Find entropy: $S = k \log \Omega$
3. Find temperature: $T^{-1} = \frac{\partial S}{\partial U}$
4. Solve for $U(T)$.
5. The heat capacity for the system is $C_V = \frac{\partial U}{\partial T}$.

Examples:

- High temperature limit of Einstein Solid:
  1. Recall that the ES has following multiplicity approximation for $n >> N$: $\Omega(N, n) \approx (\frac{en}{N})^N$.
  2. $S = k \log \Omega = Nk[1 + \log(\frac{n}{N})]$. Write $U = \epsilon n$ where ϵ is the energy level spacing.

$$S(U, V, N) = Nk \left[1 + \log\left(\frac{U}{\epsilon n}\right)\right] \text{ for } U >> N\epsilon$$

  3. $T^{-1} = \frac{\partial S}{\partial U} = \frac{Nk}{U}$
  4. Solve for U:

$$U = NkT$$

  This is the result we would have obtained by the equipartition theorem, yet we did not "enforce" the equipartition theorem; it arises naturally from the Second Law.

5. Calculate the heat capacity:

$$C_V = \frac{\partial U}{\partial T} = Nk$$

The heat capacity is a constant for this high temperature solid.

- Monatomic ideal gas: We worked this one in class starting from the Sakur Tetrode equation for the entropy. We found
  - $U = \frac{3}{2}NkT$
  - $C_V = \frac{3}{2}Nk$.

  In general, the heat capacity of an ideal gas is $C_V = (f/2)Nk$.

12.3 Paramagnetism

Recall that paramagnetism is a form of magnetism that occurs only in the presence of an externally applied magnetic field. Elements can be paramagnetic if they have unpaired electrons. Examples of paramagnetic elements include Al, Ba, O, U, Mg.

A paramagnetic material has an unpaired electron in its outer valence. This electron has a dipole moment μ. In the presence of an external magnetic field, B, the dipole can either align or anti-align with the field. The energy is increased by an amount μB for electrons counter-aligned with the external field, and decreased by $-\mu B$ for electrons aligned with field. To flip a single electron's dipole from up to down requires a total energy of $2\mu B$.

The total energy of a set of N dipoles in a field B is

$$U = \mu B(N_\downarrow - N_\uparrow)$$

Note that $U < 0$ if $N_\uparrow > N_\downarrow$ - if there are more ups than downs in the set of N dipoles. Further, define the net magnetization of the paramagnet as the quantity

$$M \equiv \mu(N_\uparrow - N_\downarrow) = -\frac{U}{B}$$

.

Now, let's carry out our prescription for calculating $U(T)$ and $C(T)$.

1. To find the multiplicity, note that this system is identical to the coin system that we've studied. So, we can straightforwardly write down the multiplicity:

$$\Omega(N, N_\uparrow) = \frac{N!}{N_\uparrow! N_\downarrow!} = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}$$

2. $S = k \log \Omega$. Apply Stirling Approximation for the case $N >> 1$:

$$\begin{aligned} \frac{S}{k} &= \log N! - \log N_\uparrow! - \log(N - N_\uparrow)! \\ &\approx N \log N - N - N_\uparrow \log N_\uparrow + N_\uparrow - (N - N_\uparrow)\log(N - N_\uparrow) + (N - N_\uparrow) \\ &\approx N \log N - N - N_\uparrow \log N_\uparrow - (N - N_\uparrow)\log(N - N_\uparrow) \end{aligned}$$

3. Now we find the temperature-energy equation of state:

$$\begin{aligned} \frac{1}{T} &= \left(\frac{\partial S}{\partial U}\right)_{N,B} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow} \\ &= -\frac{1}{2\mu B}\frac{\partial S}{\partial N_\uparrow} \\ &= \frac{k}{2\mu B}\log\left(\frac{N - U/\mu B}{N + U/\mu B}\right) \end{aligned}$$

4. The equation of state is more conventionally written as $U(T)$:

$$\Rightarrow U(N, T, B) = N\mu B\left(\frac{1 - e^{2\mu B/kT}}{1 + e^{2\mu B/kT}}\right) = -N\mu B \tanh\left(\frac{\mu B}{kT}\right)$$

The magnetization of the material is then

$$M(N,T,B) = N\mu \tanh\left(\frac{\mu B}{kT}\right)$$

5. The heat capacity C_V should be straightforward. What do you get?

The magnetization function in step (4) above is an interesting result. Experimentally, paramagnetic systems are known to obey *Curie's Law* at low values of magnetization. The temperature dependence of the magnetization when the external field B is small is given by:

$$M = C\frac{B}{T} \text{ (Curie's Law)}$$

where C is a material constant called the *Curie Constant.* Does our magnetization obey Curie's law? For small B at non-zero temperatures, we have $kT >> \mu B$ and we can Taylor expand the tanh function:

$$\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \cdots$$

. Therefore, when $kT >> \mu B$, we have

$$M(N,T,B) \approx N\mu\left(\frac{\mu B}{kT}\right)$$

Apparently, the Curie Constant is $C = N\mu^2/k$. Our paramagnetic model reproduces Curie behavior at low B.