3.2 Ideal Gas Law is An Approximation to Real Gas Behavior

The ideal gas law can be derived from the kinetic theory of gases and relies on the assumptions that (1) the gas consists of a large number of molecules, which are in random motion and obey Newton's laws of motion; (2) the volume of the molecules is negligibly small compared to the volume occupied by the gas; and (3) no forces act on the molecules except during elastic collisions of negligible duration. (Encyclopedia Brittanica). Usually these constraints limit us to studying dilute atomic or simple molecular gases at low temperatures.

3.3 Simple Applications of the IGL

How many molecules of nitrogen are in the room right now? How many oxygen molecules? The total number of molecules in the room right now is

$$N = \frac{PV}{kT} \tag{12}$$

Now make some assumptions about P, V, and T: $P = 10^5$Pa (Standard Atmospheric Pressure), $V = 5m \times 4m \times 3m = 60m^3$, and the temperature is $T = 300K$. So, $N = 1.5 \times 10^{27}$. 78% are N_2 molecules (1.1×10^{27}) and the rest (3.2×10^{26}) are O_2 molecules.

3.4 Another IGL Example Problem

Before the recent Helium Scarcity (they mine it from the Sun, right?), restaurants would freely give away helium balloons to my kids who would then free them to the sky upon leaving the restaurant. We would make a game of seeing how long we could track them before they disappeared. The thing is that, eventually, the balloons will pop due to the expansion driven by the reduced atmospheric pressure. The question is, "How high does the balloon get in the sky before it pops?" and is it possible to see it at this altitude? Have you ever seen a helium balloon pop in the sky? Me neither.

I don't really know if this is accurate or not, but let's imagine that a fully inflated helium balloon will burst when its volume increases by 10%. Further let's make tha faulty assumption that the atmospheric temperature is constant during the balloon's voyage. We can correct mistake for this later. Right now, we're just doing the spherical-cow thing, so, moo along...

Referring to the diagram on the board, let's label the initial pressure of the balloon's environment (not the pressure inside the balloon!) as P_0 and the final atmospheric pressure (where the balloon pops) as P. Let's label the balloon's initial pressure as P_1 and the balloon's final pressure as P_2. The balloon's initial and final volumes will be V_1 and V_2. Considering just the equation of state for the balloon, we have

$$P_1V_1 = NkT \text{ and } P_2V_2 = NkT \tag{13}$$

Therefore,

$$P_1V_1 = P_2V_2 \tag{14}$$

Since $V_2 = 1.1V_1$, we have

$$\frac{P_1}{P_2} = 1.1 \tag{15}$$

The result of Problem 1.16c is that the atmospheric pressure varies with height z according to

$$P(z) = P_0 e^{-\frac{mg}{kT}z} \tag{16}$$

where m is the average mass of an "air molecule":

$$m \approx 0.78m_{N_2} + 0.22m_{O_2} = [0.78(28) + 0.22(32)] \times 1.67 \times 10^{-27}kg = 4.82 \times 10^{-27}kg \tag{17}$$

The pressures must satisfy $P_0/P = P_1/P_2$. So that we have

$$1.1 = e^{\frac{mg}{kT}}z, \tag{18}$$

or $\ln(1.1) = mgz/kT$. Finally, solve for z to get the height at which the balloon bursts:

$$z = \frac{kT}{mg}\ln(1.1) = 835m \approx 2500ft \tag{19}$$

Can we see the balloon at this altitude? Recall some optics: The angular size of the balloon at this distance is given by $theta \approx \frac{}{D}$ where d is the diameter of the balloon and D is the distance from the eye. So, the angular size of a $d = 20$ cm diameter balloon in the sky when it bursts is roughly $\theta = 0.2/835 = 2.4 \times 10^{-4}$ radians or about $0.01°$.

A good rule of thumb is that the minimum angular size that we can resolve using a pupil of diameter d_{pupil} is

$$\theta_{min} = 1.22 \frac{\lambda}{d_{\text{pupil}}} \tag{20}$$

where both λ and d_{pupil} must be measured in cm. For sunlight with an average wavelength of 5000 A $= 5 \times 10^-5$ cm using an average human eye pupil of diameter 3 mm $= 0.3$ cm, we find $\theta_{min} = 1.22 \times 5.0 \times 10^{-5}/0.3 = 2.0 \times 10^{-4}$ radians. Our balloon is just under the ideal theoretical limit of resolution for the perfect pupil when it pops. So, while it is possible for the perfect eye to catch the pop, it will never be seen by the unaided, imperfect eye!

3.5 Ideal Gas Law Applied to the Solar Corona

The solar corona is a dilute gas of charged particles at a temperature of $T = 2.0 \times 10^6$K and a pressure of $P = 0.03$ Pa. What is the coronal density ($\#/m^3$). The number density is N/V. By the ideal gas law, the ratio N/V is

$$\frac{N}{V} = \frac{P}{kT} \tag{21}$$

Plugging numbers in, we have

$$\frac{N}{V} = \frac{0.03Pa}{(1.38 \times 10^{-23} J/K)(2 \times 10^6 K)} = 1.09 \times 10^{15} m^{-3} \tag{22}$$

Let's compare this to the typical number density of molecules in this room where the pressure is $P = 1atm = 1.01 \times 10^5 Pa$ and the temperature is $T = 300K$. The number density of molecules in the room is then

$$\frac{N}{V} = \frac{1.01 \times 10^5 Pa}{(1.38 \times 10^{-23} J/K)(300K)} = 2.43 \times 10^{25} m^{-3} \tag{23}$$

The particle density in this room is 10 orders of magnitude higher than in the solar corona!

4 Relation between Ideal Gas Law and Newtonian Mechanics: Equipartition Theorem

Consider a gas consisting of a single particle of mass m bounding around in a box of volume $V = L^3$. I would like to know how the temperature of this "gas" is related to the kinetic energy of the particle. Consider that the particle is initially moving with a velocity $\mathbf{v}$ with x-component v_x. The pressure exerted by the particle on the wall at $x = L$ in a collision with that wall is

$$P = \frac{F_x}{A} \tag{24}$$

with $A = L^2$ the area of the wall. By Newton's third law, the force exerted on the wall, F_x is equal and opposite to the force exerted on the particle by the wall. So, by Newton's second law, we have:

$$P = -\frac{m\frac{\Delta v_x}{\Delta t}}{A} \tag{25}$$

We are free to average the acceleration $\frac{\Delta v_x}{\Delta t}$ over any appropriate interval. Since nothing happens to particle except at the walls, let's choose the time interval Δt to correspond to the time taken by the particle in one round trip from the left wall to the right wall and back to the left wall. This time is $\Delta t = 2L/v_x$. The change in v_x in this time interval is $\Delta v_x = -2v_x$. Prove this important point!

Therefore, the average pressure exerted during one collision with the right wall is

$$\bar{P} = -\frac{m\frac{-2v_x}{(2L/v_x)}}{A} = \frac{m}{V} v_x^2 \tag{26}$$

where $V = AL$.